Beilinson-Bernstein over the base affine G/N

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October 23, 2015

Our goal is to explain some of the background used to understand the main result of [BN] and compute some examples. There is more general theory in [BN2]. Much of that theory applies to very general settings; we specialize to the case where X is a smooth scheme, and thus we use [BB] for these classical notions. There is no original work in this note, except for any mistakes, which are original and mine.

1 The theorem

1.1 Beilinson-Bernstein localization

Let Δ denote the localization functor, and Γ the global sections functor. These form an adjoint pair (Δ, Γ). Let $\mathcal{D}(G/B))_\lambda$ denote the category[1] of λ-twisted D-modules on G/B.

Theorem 1.1 (Beilinson-Bernstein). *For regular weight $\lambda \in \mathfrak{h}^*$, and corresponding class $[\lambda] \in \mathfrak{h}^*//W$, then one has an equivalence*

$$\Delta : \mathcal{U}\mathfrak{g} - \mathrm{mod}_{[\lambda]} \simeq \mathcal{D}(G/B)_\lambda : \Gamma$$

The question of what happens when we work over G/N and do not specify a monodromy is addressed in [BN]. The localization and global sections functors are still defined; however, they will not be equivalences in general. The story proceeds roughly in these steps.

1. Though the pair (Δ, Γ) are not equivalences, they are still adjoint, so one can apply the standard Barr-Beck formalism to obtain an equivalence $T - \mathrm{mod} \simeq \mathcal{U}\mathfrak{g} - \mathrm{mod}$, where $T = \Delta \circ \Gamma \in \mathrm{End}(\mathcal{D}_H(G/N))$ is the monad from the adjunction. By the right adjoint functor theorem one can find that we also have an adjunction $(\Gamma, \Delta^!)$ and a comonad $T^\vee = \Delta^! \circ \Gamma$. The next step is to find a more concrete description of the monads and their modules (resp. comonads and comodules).

2. One observes that the setup here for D-modules is "quantized" version of the Grothendieck-Springer resolution, which is Calabi-Yau, and thus one might expect, and finds, an equivalence of functors $\Delta^! \simeq \Delta^*$. So, the monad and comonad from above are equivalent endofunctors of $\mathcal{D}_H(G/N)$, and their comodules and modules are also identified.

3. Next, one wishes to explicitly identify the functor $T \simeq T^\vee$ in some way. By some general integral transform formalism in [BN2], the category of endofunctors $\mathrm{End}_{\mathcal{D}(G)}(\mathcal{P})$ is equivalent to the Hecke category $\mathcal{H} := \mathcal{D}_{H \times H}(N \backslash G/N)$. So our more concrete description of the monad will be to specify an object in this category. It turns out this object is just the sheaf of differential operators $\mathcal{D}_{N \backslash G/N}^{H \times H}$.

2 Toward the theorem

We will not be able to prove the theorem in its entirety in this talk. We will instead opt to give a general sense of the formalism, and to do a few examples.

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[1]This should probably be some kind of dg-category or ∞-category, but I'll relegate this to the growing pile of technical details I won't treat here.

2.1 Equivariant D-modules

Remark 2.1 (G-equivariant D-modules on an G-torsor). Suppose that G acts freely on P; then $p : P \to B = P/G$ is a G-torsor. Here we will describe the two kinds of G-equivariant D-modules on P, corresponding statements on B (where we've "killed" the G-action), and corresponding "quantization" heuristics.

on total space P with G-action	on the base $B = P/G$	classical limit
weak G-equivariant D-module	$(p_*\mathcal{D}_P)^G$-module	module on $(T^*P)/G$
strong G-equivariant D-module	$\mathcal{D}_B$-module	module on $T^*(P/G) = T^*(B)$

The equivalence of the first two columns can be made precise as an equivalence of categories (see [BB] 1.8.9, 1.8.7). The third column is more of a heuristic, since one still has to specify the quantization.

Definition 2.1. Now for some actual definitions.[2] Let G act on a smooth variety X. Then $\mathcal{D}_X$ inherits a natural G-action. Let us be a little more precise about what extra structure this provides. We have (1) a left and right action of $\mathfrak{g}$ on $\mathcal{D}_X$ coming from the natural action on $\mathcal{T}_X$, (2) considering $\mathcal{D}_X$ as a left or right $\mathcal{O}_X$-module, the structure of a G-equivariant sheaf, also induced by the equivariant structure on $\mathcal{T}_X$, and (3) a compatibility condition that the adjoint action of G on $\mathfrak{g}$ intertwines with the actions, which is evidently satisfied by the construction.

A *weak equivariant* (left) D-module is a (left) D-module M which is given the equivariant structure of a (left) $\mathcal{O}_X$-module, satisfying the compatibility condition with the equivariant structure on $\mathcal{D}_X$, i.e. that $g^*(dm) = g^*(d)g^*(m)$ for $d \in \mathcal{D}_X$ and $m \in M$. A *strong equivariant* D-module is a weak equivariant D-module such that the induced action of $\mathfrak{g}$ on M agrees with the action of the image of $\mathfrak{g}$ in $\mathcal{D}_X$.

Remark 2.2 (On fibers of vector bundles with flat connection). Say G acts on X, and suppose $\mathcal{M}$ is a $\mathcal{D}_X$-module which is a flat connection (i.e. coherent as an $\mathcal{O}_X$-module). Giving $\mathcal{M}$ a G-equivariant structure (as an $\mathcal{O}_X$-module) structure is to give isomorphisms $M_x \simeq M_{g \cdot x}$ for every $(g, x) \in G \times X$, satisfying some compatibility condition. A $\mathcal{O}_X$-coherent D-module has an integrable connection, so one can integrate the differential equations to also obtain isomorphisms $M_x \simeq M_{g \cdot x}$. In the weak case, these are not required to agree. In the strong case, they are.

Remark 2.3 (Notation). We will use $\mathcal{D}(X/G)$ to denote the category of strictly G-equivariant D-modules on X, and $\mathcal{D}_G(X)$ to denote the category of weakly G-equivariant D-modules on X.

Example 2.1 (H-equivariant/monodromic D-modules on H). Let $H = \mathbb{G}_m$ and let us consider the basic example of H-equivariant/monodromic D-modules on H (under the left multiplication action). We expect to find that (1) weakly H-equivariant D-modules on H are $\Gamma(H, \mathcal{D}_H)^H$-modules are noncommutative modules on $T^*(H)/H = (H \times \mathfrak{h}^*)/H = \mathfrak{h}^*$, i.e. $\mathcal{U}\mathfrak{h}^*$-modules, and (2) strongly H-equivariant D-modules on H correspond to D-modules on a point, i.e. vector spaces.

Let $H = \operatorname{Spec} k[x, x^{-1}]$. Then $\mathcal{D}_H = k\langle x, x^{-1}, \partial_x \rangle$ with relations $[\partial_x, x] = 1$ and $[\partial_x, x^{-1}] = x^{-2}$. A $\mathbb{G}_m$-action is the same as a weight grading, and we have $\operatorname{wt}(x) = 1$ and $\operatorname{wt}(\partial_x) = -1$. Thus $(\mathcal{D}_H)^H = k[x\partial_x] \simeq \operatorname{Sym}^\bullet(\mathfrak{h}^*)$, since $x\partial_x$ is the vector field which generates the action of H.

Thinking in terms of vector bundles, a weakly H-equivariant D-module has two isomorphisms of fibers: by equivariance, and by parallel transport. In this case, the equivariant structure provides us with distinguished isomorphisms of fibers of a vector bundle M on H, and adding parallel transport gives us a monodromy for the generator of the cyclic group, i.e. a $U(\mathfrak{h}^*)$-module structure on a fiber. For strongly equivariant D-modules these two notions must coincide, so there isn't any extra structure, and the entire module is determined by any fiber.

Example 2.2 (H-monodromic D-modules on G/N for $G = SL_2$). One can find explicitly that $\mathcal{D}_H(G/N)$ is globally generated, and that

$$\mathcal{D}_H(G/N) = (p_*\mathcal{D}_{G/N})^H \simeq \mathcal{U}\mathfrak{g} \otimes_{\mathcal{Z}\mathfrak{g}} \mathcal{U}\mathfrak{h}$$

In light of the above example, roughly this means that one has the usual generating vector fields on the flag variety G/B but also extra vector fields which generate the H-action on the torsor G/N.

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[2]In [BN2], D-modules are defined in a "dual" way as ind-coherent sheaves on the deRahm stack (also known as algebraic crystals). In the 2011 version of the same paper, they are defined in the "classical" way as we have done here, and the two coincide when X is smooth. This is a story worth telling, but I don't think I have time for it in this talk.

This can be worked out explicitly for SL_2 fairly directly. Take B to be upper triangular matrices, and N strictly upper triangular matrices. Take coordinates $(s,t) = \begin{pmatrix} s & * \\ t & * \end{pmatrix} N$ on $\mathbb{A}^2 - 0 = G/N$. One can compute using the G-action that $E = t\partial_s$, $F = s\partial_t$ and $H = s\partial_s - t\partial_t$.[3] To compute $\mathcal{D}_H(G/N)$, note that $\deg(s,t) = 1$ and $\deg(\partial_s, \partial_t) = -1$. If we take the open affine where $s \neq 0$, then one as generating vector fields $s\partial_s, t\partial_t$ over the ring $k[t/s]$. So one finds that the global vector fields in $\mathcal{D}_H(G/N)$ are $H = s\partial_s - t\partial_t, E = t\partial_s, F = s\partial_t$ and the central "monodromy" vector field $s\partial_s + t\partial_t$ which generates the H-action.

Remark 2.4 (Specifying monodromy)**.** When H is a torus, acting freely on a space, then $\mathcal{D}_H(X)$ will contain $U(\mathfrak{h})$ and thus we can tensor with characters of H to specify a monodromy. Thus in the above example, note that specifying a monodromy for weight λ gives

$$\mathcal{U}\mathfrak{g} \otimes_{\mathcal{Z}\mathfrak{g}} \mathcal{U}\mathfrak{h} \otimes_{\mathcal{U}\mathfrak{h}} k_\lambda = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{Z}\mathfrak{g}} k_{[\lambda]}$$

bringing us back to the original setup for Beilinson-Bernstein.

2.2 Barr-Beck

Let **B** and **E** be some kind of categories: a 1-category, or an ∞-category.[4] Consider adjoint functors (I, R):[5]

$$\mathbf{B} \underset{R}{\overset{I}{\rightleftarrows}} \mathbf{E}$$

Theorem 2.1 (Barr-Beck for monads)**.** *Let $R : \mathbf{E} \to \mathbf{B}$ be a functor with left adjoint I, and $T = RI$ a monad on* **B***. Suppose that* **E** *has homotopy colimits of R-split cosimplicial objects (cosimplicial objects whose image under R has a split homotopy colimit), and R preserves and reflects homotopy colimits of such cosimplicial objects. Then we have an equivalence (factoring the original functors):*

$$\mathrm{Mod}_T(\mathbf{B}) \rightleftarrows \mathbf{E}$$

Theorem 2.2 (Barr-Beck for comonads)**.** *Let $I : \mathbf{B} \to \mathbf{E}$ be a functor with right adjoint R, and $T = IR$ a comonad on* **E***. Suppose that* **B** *has homotopy limits of I-split simplicial objects (simplicial objects whose image under I has a split homotopy limit), and I preserves and reflects homotopy limits of such simplicial objects. Then we have an equivalence:*

$$\mathbf{B} \rightleftarrows \mathrm{Comod}_T(\mathbf{E})$$

In some sense Barr-Beck tells us how a pair of adjoint functors fail to be an equivalence, and gives a purely abstract-nonsense characterization for what structure must be furnished to one of the categories to make them equivalent. A recipe for doing this kind of analysis might be: apply Barr-Beck, and then interpret what it means to be a T-module (or comodule).

2.2.1 Example: monoids

Let us work through Barr-Beck (for 1-categories) for the free and forgetful functors between sets and monoids.[6] In this case, the monoid T is the "words" functor: it takes a set and returns the set of all words (including the empty word). A module structure on a set A is a "structure map" of sets $T(A) \to A$. Consider the simplicial diagram:

$$T^3(A) \Rrightarrow T^2(A) \rightrightarrows T(A) \longrightarrow A$$

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[3]Note E becomes a negative weight.

[4]We will frequently play fast-and-loose with such technical details in this section in an attempt to appeal to the need for (1) easy concrete examples and (2) a conceptual setting in which Barr-Beck looks most naturally at home.

[5]The choice of lettering is motivated as follows: I stands for induce, R stands for restrict, **B** stands for base category, and **E** stands for enriched category.

[6]I work with monoids out of pure laziness. One could easily do the same for groups, but I don't want to have to say "inverses."

The object $T^2(A)$ consists of words of words, or words with two levels of nesting. The two maps $T^2(X) \to T(X)$ consist of concatenating the words on each level. For example, if one has $(3 + 4 + 5) + (3) + (4 + 5)$, one map willl send this to $12 + 3 + 9$ and another will send it to $3 + 4 + 5 + 3 + 4 + 5$. That the diagram is a coequalizer is exactly associativity of the monoid operation. The $T^3(A)$ level records "higher associativities" that don't come into play here, but would in contexts that require the full machinery of ∞-categories.

2.2.2 Example: descent on BG

Let G be an affine algebraic group. Let $f : \mathrm{pt} \to \mathrm{pt}/G = BG$. The comonad $T^\vee = f^* f_*$ can be understood by base change on the diagram

$$
\begin{array}{ccc}
G & \longrightarrow & \mathrm{pt} \\
\downarrow & & \downarrow \\
\mathrm{pt} & \longrightarrow & BG
\end{array}
$$

Thus $T^\vee(V) = V \otimes k[G]$, and a $T^\vee$-comodule is the structure map

$$V \to V \otimes k[G]$$

The coherence relations for comodules translates to the usual coherence relations for comodules, and thus one finds that $\mathrm{QCoh}(BG)$ is equivalent to $k[G]$-comodules, or rational G-representations.

2.3 Quantizing the Grothendieck-Springer resolution

Consider the classical picture, where we take (stacky quotients)

$$\mathcal{B} = G/B, \quad \tilde{\mathcal{B}} = G/N, \quad Z = B\backslash G/B, \quad \tilde{Z} = N\backslash G/N.$$

One has

$$T^*(\tilde{\mathcal{B}}) = \{(gN, x) \in G/N \times \mathfrak{g}^* \mid x \in (g\mathfrak{n}g^{-1})^\perp \simeq \mathfrak{b}\}$$

and observing that $\tilde{\mathfrak{g}} \simeq T^*(\tilde{\mathcal{B}})/H$[7], the Grothendieck-Springer resolution $\tilde{\mathfrak{g}} \to \mathfrak{g}$ can be understood as a map

$$\mu : T^*(\tilde{\mathcal{B}})/H \to \mathfrak{g}$$

where $H = B/N$ is the universal Cartan, and the in the quotient H acts on the "base" of the vector bundle $T^*(\tilde{\mathcal{B}})$.

Remark 2.5 (Calabi-Yau)**.** The bundle $\tilde{\mathfrak{g}}$ is Calabi-Yau. This follows from the fact that if E is a vector bundle over X with sheaf of sections $\mathcal{E}$, then[8]

$$\omega_E \simeq \pi^*(\omega_X \otimes \overset{\mathrm{top}}{\bigwedge}(\mathcal{E}^\vee))$$

In our case, $\tilde{\mathfrak{g}}$ is a extension by trivial bundles of $\mathcal{N}$, so it suffices to prove the statement for $\mathcal{N}$. But $\mathcal{N}$ has sheaf of sections Ω^1_X, whose top power is ω_X.

Remark 2.6 (Serre duality)**.** For sufficiently reasonable spaces, the adjoint functor theorem tells us that the functor f_* of quasicoherent sheaves has a right adjoint, which we write $f^!$. In the case that X is smooth of dimension n, and $f : X \to \mathrm{pt}$, $f^!(k) = \omega_X[n]$, and one has

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega)^\vee$$

One can define a "Serre duality" functor

$$\mathbb{D}(\mathcal{F}) = R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$$

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[7]The isomorphism is given by $(gB, x) \mapsto H \cdot (gN, x)$.

[8]Take the top exterior power of the short exact sequence $0 \to \pi^*\Omega^1_X \to \Omega^1_E \to \Omega^1_{E/X} \to 0$ and check that $\Omega^1_E = \mathcal{E}^\vee$.

and observe that it is a contravariant equivalence of categories with $\mathbb{D}^2 = 1$, and that $f^! = \mathbb{D}f^*\mathbb{D}$.

Remark 2.7 (Verdier duality)**.** In the D-modules setting, one can define

$$\mathbb{D}(\mathcal{M}) = R\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}[n]$$

Note that the Hom functor turns a left D-module $\mathcal{M}$ into a right D-module, and that tensoring with ω_X^{-1} turns a right D-module into a left D-module. The details are somewhat technical and I won't go into them here. However, using this, one can define a functor $f^! = \mathbb{D} \circ f^* \circ \mathbb{D}$ which is a right adjoint to f_* (which one can see purely by adjoint functor calculus).

Remark 2.8. Suppose that $f : X \to Y$ is a map of smooth schemes, where Y is affine and X is Calabi-Yau. Then we claim that $f^* = f^!$. Since all dualizing sheaves are trivial, this is just the claim that $(f^*M^\vee)^\vee = f^*M$, but since everything is smooth, and Y is affine, all coherent sheaves M have free resolutions whose pullbacks are also free resolutions.

Remark 2.9 (Quantized realization of the Calabi-Yau property)**.** The quantized version of the adjoint functors (μ^*, μ_*) and $(\mu_*, \mu^!)$ are

$$\begin{array}{ccc}
\gamma^* : \mathcal{U}\mathfrak{g} - \mathrm{mod} \to \mathcal{D}_H(G/N) & \gamma_* : \mathcal{D}_H(G/N) \to \mathcal{U}\mathfrak{g} - \mathrm{mod} & \gamma^! : \mathfrak{g} - \mathrm{mod} \to \mathcal{D}_H(G/N) \\
M \mapsto \mathcal{D}_{G/N} \otimes_{\mathcal{U}\mathfrak{g}} M & \mathcal{M} \mapsto \Gamma(G/N, \mathcal{M}) = \mathrm{Hom}_{\mathcal{D}_{G/N}}(\mathcal{D}_{G/N}, \mathcal{M}) & M \mapsto \mathbb{D} \circ \gamma^* \circ \mathbb{D}
\end{array}$$

where $\mathbb{D}$ on $\mathcal{U}\mathfrak{g} - \mathrm{mod}$ is defined by

$$\mathbb{D}(M) = R\,\mathrm{Hom}_{\mathcal{U}\mathfrak{g}}(M, \mathcal{U}\mathfrak{g}[\dim(\mathfrak{g})])$$

Because $\tilde{\mathfrak{g}}$ is Calabi-Yau, we find that $\gamma^* = \gamma^!$. Thus, the monad $T = \gamma^!\gamma_*$ and the comonad $T^\vee = \gamma^*\gamma_*$ endofunctors of $\mathcal{D}_H(G/N)$ are in fact equal, so Barr-Beck gives equivalences

$$T - \mathrm{mod} \simeq \mathcal{U}\mathfrak{g} - \mathrm{mod} \simeq T^\vee - \mathrm{comod}$$

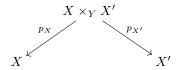We will interpret these monads and comonads in the next section.

2.4 Integral transforms

2.4.1 In classical settings

Theorem 2.3 ([BFN] Theorem 1.2)**.** *Let $X \to Y$ and $X \to Y'$ be maps of perfect stacks (e.g. quasicompact derived schemes with affine diagonal, X/G in characteristic zero for G affine). Then there is a canonical equivalence*

$$\mathrm{QCoh}(X \times_Y X') \to \mathrm{Fun}_{\mathrm{QCoh}(Y)}(\mathrm{QCoh}(X), \mathrm{QCoh}(X'))$$

The equivalence is realized by *integral transforms*. That is, for an integral kernel $\mathcal{K} \in \mathrm{QCoh}(X \times_Y X')$, one defines a functor $F_\mathcal{K} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X')$ by convolution:

i.e. the formula $F_\mathcal{K} = (p_{X'})_*(p_X^* - \otimes\mathcal{K})$. For example, the structure sheaf of the diagonal $\Delta \subset X \times_Y X$ corresponds under this equivalence to the identity functor. For a map $f : X' \to X$ over Y, the pullback of f corresponds to taking the structure sheaf of the graph $\Gamma_f \subset X \times_Y X'$.
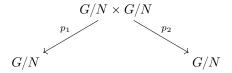
2.4.2 In microlocal settings

This section will contain very few proofs, because many of the proofs rely on technical results on D-modules. I will only try to communicate the broad ideas, with a general appeal to analogy with the classical (non-quantized) situation. The following result is analogous to the classical one, but its conditions are much more restrictive.

Theorem 2.4 ([BN2], Theorem 1.14). *Let $X_1 \to Y$ be a Deligne-Mumford stack over a scheme Y[9], and $X_2 \to Y$ an arbitrary stack. Then the natural maps are equivalences[10]*

$$\mathcal{D}(X_1) \otimes_{\mathcal{D}(Y)} \mathcal{D}(X_2) \to \mathcal{D}(X_1 \times_Y X_2) \to \mathrm{Fun}^L_{\mathcal{D}(Y)}(\mathcal{D}(X_1), \mathcal{D}(X_2))$$

Let us describe how this result applies to our current situation. Barr-Beck tells us that we are interested in monads and comonads corresponding to the adjunction (γ^*, γ_*) and $(\gamma_*, \gamma^!)$, which are actually naturally equivalent by a Calabi-Yau property. We want to explicitly identify this functor by some sheaf.

$$
\begin{array}{ccc}
 & G/N \times G/N & \\
{\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\
G/N & & G/N
\end{array}
$$

A monodromic version of the result above (see [BN2] Section 7) tells us that

$$\mathcal{D}_{H \times H}(G/N \times G/N) \simeq \mathrm{End}^L(\mathcal{D}_H(G/N))$$

and so our endofunctor of interest $T = T^\vee$ arises by an integral transform $(p_2)_*((p_1)^* - \otimes \mathcal{W})$[11] for some kernel $\mathcal{W} \in \mathcal{D}(G/N \times G/N)$ such that

$$T = (p_2)_*((p_1)^! - \otimes \mathcal{W}) = (p_2)_*((p_1)^! \otimes \mathcal{W}) = T^\vee$$

A quantized version of base change on the diagram

$$
\begin{array}{ccc}
\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} & \longrightarrow & \tilde{\mathfrak{g}} \\
\downarrow & & \downarrow \\
\tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g}
\end{array}
$$

tells us that $(p_2)_*(p_1)^! = \gamma^! \gamma_* = T$ and likewise for star pullback. Thus, comparing

$$T = \gamma^! \gamma_*$$

$$F_{\mathcal{W}} = (p_2)_*((p_1)^! - \otimes \mathcal{W})$$

suggests that tensoring with $\mathcal{W}$ should be some quantum analogue of tensoring with the structure sheaf (i.e. "doing nothing").

Definition 2.2. The *universal Weyl sheaf* $\mathcal{W} \in \mathcal{D}_{H \times H}(\tilde{Z})$ is the sheaf of differential operators on $\tilde{Z}$ with its canonical $H \times H$ weak equivariant structure.

The result quoted above is actually stronger than what we have shown. Namely, $\mathcal{W}$ is a D-module on $N \backslash G/N$, not $G/N \times G/N$, but it comes from a G-equivariant D-module on $G/N \times G/N$. This is the story we've omitted: the category $\mathcal{D}_H(G/N)$ carries a $\mathcal{D}(G)$-action which categorifies the left G-action on G/N, and all the functors we

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[9]A Deligne-Mumford stack is a sheaf of groupoids which admits an etale cover by algebraic spaces, and whose diagonal map is representable by algebraic spaces.

[10]The L denotes functors which have left adjoints, equivalently, continuous.

[11]Note that here we don't mix left and right adjoints, unlike in the classical setting. We use only right adjoints, and so the resulting functor is also a right adjoint.

defined are linear with respect to this action by tensoring with modules in $\mathcal{D}(G)$. On the integral kernel side, this translates to integral kernels having a strong G-equivariant structure. The theorem reflecting this equivariance is:

Theorem 2.5 ([BN] Theorem 3.9). *There is an equivalence*

$$\mathcal{D}_{H \times H}(N\backslash G/N) \simeq \mathrm{End}_{\mathcal{D}(G)}(\mathcal{D}_H(G/N))$$

2.5 Example of Weyl sheaf for $G = SL_2$

Now, one can do geometric computations on these functors. What follows is a demonstration of the kind of calculus that can be done.

- The universal Weyl sheaf is a $H \times H$-monodromic D-module on $N\backslash G/N$. We will realize it as a strictly left N-equivariant, weakly $H \times H$-equivariant D-module on G/N. It is a bit complicated to consider all this structure at once, so let's make some simplifications. Recall throughout that $\mathcal{W}$ is some kind of stand-in for a endofunctor $\mathcal{D}_H(G/N) \to \mathcal{D}_H(G/N)$.

- First, let us impose the strict left N-equivariance. What this means is that two different (left) actions of $\mathfrak{n} = \langle E = t\partial_s \rangle$ on $\mathcal{D}_{G/N}$ should coincide. The first is the action one gets by differentiating the N-action on $\mathcal{D}_{G/N}$ as a left $\mathcal{O}_X$-module;

  The second is by inclusion of the vector field $t\partial_s$ into $\mathcal{D}_{G/N}$, which comes from differentiating the action of G on X.

  In particular, E acts on E by zero (adjoint), so E should be sent to zero.

  By Hamiltonian reduction, this means we impose the equation $t\partial_s = 0$.

- Next, fix a right H-monoromy $\lambda \in \mathfrak{h}^*$, and forget entirely the left H-equivariance. This means we are computing the λ-monodromic component in the target category, and the Weyl sheaf lives in $\mathcal{D}_\lambda(N\backslash B/N)$, representing a functor $\mathcal{D}(G/N) \to \mathcal{D}_\lambda(G/N)_\lambda$. The perscribed monodromy gives us an equation $s\partial_s + t\partial_t = \lambda$.

- Now, we have that

$$\mathcal{W} = \mathcal{D}_{G/N}/\langle t\partial_s, s\partial_s + t\partial_t - \lambda \rangle$$

  Note the singular support condition: if $t = 0$ and $s \neq 0$, then $\xi_s = 0$. If $t \neq 0$, then $\partial_s = 0$ and consequently also $\partial_t = 0$. Thus, the singular support of $\mathcal{W}$ is the conormal to the closures of the stratification of G/N given by:

$$j : U = \mathbb{A}^1 \times \mathbb{G}_m \to G/N \qquad i : V = \mathbb{G}_m \times \{0\} \to G/N$$

  Also note that these are preimages of the left N-orbits of G/B. This is a λ-monodromic D-module can be studied as a "twisted" $(\pi_* \mathcal{D}_{G/N})^\lambda$ D-module on G/B, or a $\mathcal{D}_{G/N}^\lambda$-module on G/N.

- Let us pull back to the line $s = 1$. Describing the D-module on this line is analogous to describing it on an open affine $\mathbb{A}^1 \subset \mathbb{P}^1 = G/B$; by its singular support, we know that it must be a flat connection at infinity, so $\mathcal{W}_\lambda$ is determined by its behavior on this line. Setting $s = 1$, we find that $\partial_s = \lambda - \partial_t$ and $t\partial_s = 0$, so we have

$$\mathcal{W}_{\mathbb{A}^1, \lambda} = \frac{\mathcal{D}_{\mathbb{A}^1}}{\mathcal{D}_{\mathbb{A}^1}(t^2\partial_t - \lambda t)} = \frac{\mathcal{D}_{\mathbb{A}^1}}{\mathcal{D}_{\mathbb{A}^1}(t\partial_t t - (\lambda+1)t)} = \frac{\mathcal{D}_{\mathbb{A}^1}}{\mathcal{D}_{\mathbb{A}^1}(\partial_t t^2 - (\lambda+2)t)}$$

This D-module sits in short exact sequences:

$$0 \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow \mathcal{D}/\mathcal{D}(t^2\partial_t - t\lambda) \longrightarrow \mathcal{D}/\mathcal{D}(t\partial_t - \lambda) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow \mathcal{D}/\mathcal{D}(t\partial_t t - (\lambda+1)t) \longrightarrow \mathcal{D}/\mathcal{D}(\partial_t t - (\lambda+1)) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{D}/\mathcal{D}(t\partial_t - (\lambda+1)) \longrightarrow \mathcal{D}/\mathcal{D}(t\partial_t t - (\lambda+1)t) \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow 0$$

$$0 \longrightarrow \mathcal{D}/\mathcal{D}(\partial_t t - (\lambda+2)) \longrightarrow \mathcal{D}/\mathcal{D}(\partial_t t^2 - (\lambda+2)t) \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow 0$$

We will investigate, for various λ, whether the sequence splits, and also identify the components.

- The component on the left is known and does not depend on λ: it is the pushforward $i_*\mathcal{O}_V$. Explicitly, this pushforward needs to be a D-module supported at zero, and on which t acts by zero. Since $\mathcal{O}_V$ is holonomic, and $\partial t - t\partial$ acts by $-t\partial$ acts by 1, we find that $\mathcal{O}_V = k[\partial]$ as a vector space, with t acting on ∂^n by $(-1)^n n!$.

- Let us investigate the term on the right. First I claim that

$$j_*\mathcal{O}_U = D/\partial t \qquad j_!\mathcal{O}_U = D/t\partial$$

To compute $j_*\mathcal{O}_U = \mathcal{O}_U \otimes_{\mathcal{D}_U} \mathcal{D}_{\mathbb{A}^1}$, we want to find a $\mathcal{D}_{\mathbb{A}^1}$-generator for

$$\mathcal{O}_U = \frac{k\langle t, t^{-1}, \partial_t \rangle}{\mathcal{D}_U \partial_t}$$

and then find a defining differential equation on $\mathbb{A}^1$. Since $\partial_t \cdot t^{-1} = -t^{-2}$, we can get all positive powers of t by multiplication by t and all negative powers by differentiation ∂_t. Thus, as a $\mathcal{D}_{\mathbb{A}^1}$-module, t^{-1} is a generator for $j_*\mathcal{O}_U$ satisfying the differential equation $\partial_t t = 0$, and so $j_*\mathcal{O}_U = D/\partial t$.

For the shriek extension, note that Verdier duality just takes the tranpose of the relations, so $\mathcal{O}_U$ is Verdier self dual, and $j_!\mathcal{O}_U = \mathbb{D}j_*\mathbb{D}\mathcal{O}_U = D/t\partial$.

- Consider the following table.

$$
\begin{array}{cccc|cccc}
\cdots & D/t\partial - 1 & D/t\partial & \big| & D/t\partial + 1 & D/t\partial + 2 & \cdots \\
\hline
\cdots & D/\partial t - 2 & D/\partial t - 1 & \big| & D/\partial t & D/\partial t + 1 & \cdots
\end{array}
$$

We claim that everything to the left of the vertical line is isomorphic to $j_!\mathcal{O}_U$ and everything to the right is isomorphic to $j_*\mathcal{O}_U$. It's easy to see that the modules in a given column are isomorphic just by using the commutating relation. For the horizontal equivalences, verify that the following chain of maps are well-defined

$$D/t\partial + 1 \xleftarrow{\;\partial\;} D/t\partial + 2 \xleftarrow{\;\frac{1}{2}\partial\;} D/t\partial + 3$$

and have inverses given by

$$D/t\partial + 1 \xrightarrow{\;x\;} D/t\partial + 2 \xrightarrow{\;x\;} D/t\partial + 3$$

so that the right side of the line are all isomorphic. This is motivated by the idea that we can symbolically write $1 \in D/t\partial + n$ by the function x^{-n}, and all the D-module operations "make sense" in this notation.[12] We find that this tells us that everything on the right are isomorphic. Now note that the Fourier transform switches $D/x\partial + n$ with $D/\partial x - n$ and so modules on the left side are isomorphic as well.

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[12]It becomes difficult to interpret the D-module $D/t\partial$ in this way, since its solution isn't given by a function.

To see that $j_*\mathcal{O}_U \not\simeq j_!\mathcal{O}_U$, note the short exact sequences

$$0 \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow \mathcal{D}/\mathcal{D}t\partial \longrightarrow \mathcal{D}/\mathcal{D}\partial \longrightarrow 0$$

$$0 \longrightarrow \mathcal{D}/\mathcal{D}\partial \longrightarrow \mathcal{D}/\mathcal{D}\partial t \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow 0$$

It's not hard to see that the D-modules on the ends are simple, and that the sequence cannot split since the quotient doesn't have a lift as a submodule. For example, in $\mathcal{D}/\mathcal{D}\partial t$, we can symbolically replace 1 with x^{-1}, so elements in this D-module are Laurent polynomials; the submodule is (in these symbols) given by polynomial functions. But multiplying a Laurent polynomial by by x^N for sufficiently large N will always land us in the submodule consisting of polynomials, so the quotient cannot have a lift as a submodule.

Further, now take $M_\lambda = D/D(x\partial - \lambda)$ where λ is non-integral. The same argument above applies; for non-integral λ, μ, we have that $M_\lambda \simeq M_\mu$ if $\lambda - \mu \in \mathbb{Z}$. The converse is also true; $t\partial$ acts semisimply on these modules, so we can decompose it into its weight spaces, which must lie in the integer lattice $\lambda + \mathbb{Z}$.

- The upshot of this is these isomorphisms is that we have an easy description of splittings in the short exact sequences. Note that in the middle two short exact sequences, $\mathcal{D}/\mathcal{D}(\partial t - (\lambda + 1)) \simeq \mathcal{D}/\mathcal{D}(t\partial - (\lambda + 1))$ if and only if $\lambda \neq -1$. One can check that twisting by this isomorphism, the two short exact sequences are splittings of each other, but only when $\lambda \neq -1$. When $\lambda = -1$, the short exact sequences read:

$$0 \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow \mathcal{D}/\mathcal{D}t\partial t \longrightarrow \mathcal{D}/\mathcal{D}\partial t \longrightarrow 0$$

$$0 \longrightarrow \mathcal{D}/\mathcal{D}t\partial \longrightarrow \mathcal{D}/\mathcal{D}t\partial t \longrightarrow \mathcal{D}/\mathcal{D}t \longrightarrow 0$$

A splitting would show $\mathcal{D}/\mathcal{D}t\partial \simeq \mathcal{D}/\mathcal{D}\partial t$, which we've already ruled out.

- To summarize:

$$\mathcal{W}_{\mathbb{A}^1, \lambda} = \begin{cases} j_{*/!}\mathcal{O}_{U,\lambda} \oplus i_*\mathcal{O}_V & \lambda \notin \mathbb{Z} \\ j_*\mathcal{O}_U \oplus i_*\mathcal{O}_V & \lambda = 0, 1, 2, \ldots \\ j_!\mathcal{O}_U \oplus i_*\mathcal{O}_V & \lambda = -2, -3, -4, \ldots \\ \mathcal{T} = D/\partial t\partial & \lambda = -1 \end{cases}$$

- When we restricted to $\mathbb{A}^1$ we destroyed some monodromy data, but using this, one computes the following. Let

$$\mathcal{L}_{V,\lambda} = \frac{\mathcal{D}_V}{\mathcal{D}_V(s\partial_s - \lambda)} \qquad \mathcal{L}_{U,\lambda} = \frac{\mathcal{D}_U}{\mathcal{D}_U(\partial_s, t\partial_t - \lambda)}$$

The first is obtained by setting $t = 0$ in $\mathcal{W}_\lambda$, the latter by inverting t. Then we get:

$$\mathcal{W}_\lambda = \begin{cases} j_{*/!}\mathcal{L}_{U,\lambda} \oplus i_*\mathcal{L}_{V,\lambda} & \lambda \notin \mathbb{Z} \\ j_*\mathcal{L}_{U,\lambda} \oplus i_*\mathcal{L}_{V,\lambda} & \lambda = 0, 1, 2, \ldots \\ j_!\mathcal{L}_{U,\lambda} \oplus i_*\mathcal{L}_{V,\lambda} & \lambda = -2, -3, -4, \ldots \\ \mathcal{T} = \mathcal{D}/\mathcal{D}(t\partial_s, d\partial_s + t\partial_t) & \lambda = -1 \end{cases}$$

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