Algebraic representations of the circle

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0.0.1. We are interested in the following examples of groups and their representation theory.

- $G \subset GL_n$ a classical affine algebraic group scheme,
- G a coaffine group stack, such as $G = B\mathbb{G}_a$,
- G an animated group, such as $G = S^1$,
- G an abelian variety such as $G = E$ an elliptic curve.

0.0.2. We want to discuss two notions.

- Representation theory cannot see non-affineness of groups, so the representation theory of $B\mathbb{G}_a$, S^1 and E all coincide. But their categorical representation theories diverge.
- The trivial representation knows everything about the representation theory of unipotent groups, but far from it for reductive groups. In categorical representation theory, the trivial representation knows everything about reductive groups as well. But it does not know everything about the categorical representation theory of S^1.

This is spelled out in the following table.[1]

	G 0-affine? affinization?	BG 0-affine?	G 1-affine?	BG 1-affine?
G affine sch.	yes	yes if G unipotent no otherwise	yes	yes
G coaffine st.	yes	yes	yes	yes
G animated	no $\operatorname{Spec} \mathcal{O}(G)$ coaffine	no	yes	no
G abelian var.	no $\operatorname{Spec} \mathcal{O}(G)$ coaffine	no	yes	???
$G = B\mathbb{G}_a$	yes	yes	yes	yes
$G = S^1$	no $\operatorname{Spec} \mathcal{O}(G) = B\mathbb{G}_a$	no	yes	no
$G = E$	no $\operatorname{Spec} \mathcal{O}(G) = B\mathbb{G}_a$	no	yes	???
	$\operatorname{Rep}(G)$ only knows $\operatorname{Aff}(G)$	"Koszul duality"	$2\operatorname{Rep}(G)$ only knows $1\operatorname{Aff}(G)$	"equiv'n corresp."

0.0.3. The notion of 1-affineness was initially studied by Gaitsgory.

[1] Often, up to renormalization. E.g. $B^4\mathbb{G}_a$ is not 1-affine, but I suspect this is just a renormalization issue?

1 0-categorical

1.0.1. We define the category of representations in the usual way.

Definition 1.0.2. Let G be a group algebra in prestacks over k. Then, $\mathcal{O}(G)$ is a coalgebra object in $\mathbf{Vect}_k$ and we define

$$\mathrm{Rep}(G) := \mathrm{Mod}_{\mathbf{Vect}_k}(\mathcal{O}(G)).$$

It is a general fact that

$$\mathrm{Rep}(G) \simeq \mathrm{QCoh}(BG).$$

1.0.3. Our first observation is that $\mathrm{Rep}(G)$ cannot distinguish between G and its (0-)affinization $\mathrm{Spec}\,\mathcal{O}(G)$. So the study of $\mathrm{Rep}(G)$ is the same as the study of $\mathrm{Aff}(G)$.

1.0.4. Our next observation is that sometimes, the category $\mathrm{Rep}(G)$ is "affine." Let k denote the trivial representation, and note that $\mathrm{End}_G(k,k) \simeq \mathcal{O}(BG)$. We have commuting adjoint functors

$$\begin{array}{ccc} \mathrm{Rep}(G) & \xrightarrow{\quad\simeq\quad} & \mathrm{QCoh}(BG) \\ \Big\| & & {\scriptstyle -\otimes_{\mathcal{O}(BG)}\mathcal{O}_{BG}}\uparrow\downarrow{\scriptstyle \Gamma} \\ \mathrm{Comod}(\mathcal{O}(G)) & \underset{(-)^G := \mathrm{Hom}_G(k,-)}{\overset{-\otimes_{\mathcal{O}(BG)} k}{\leftrightarrows}} & \mathrm{Mod}(\mathcal{O}(BG)) \end{array}$$

We say a group G has a 0-equivariantization correspondence if the adjunction above defines inverse equivalences, possibly up to renormalization.[2] This is closely related to the (0-)affineness of BG.

1.0.5. Let us check this in examples.

- If G is a reductive affine algebraic group, then obviously $(-)^G : \mathrm{Rep}(G) \to \mathrm{Mod}(k)$ is not an equivalence. If G is unipotent, say $G = B\mathbb{G}_a$, then $\mathrm{Rep}(G)$ is generated by the trivial representation, and $\mathrm{End}_{\mathbb{G}_a}(k,k) \simeq k[\eta]$ where $\eta \in \mathrm{Ext}^1_{\mathbb{G}_a}(k,k)$, so we have equivalences

$$\mathrm{Rep}(\mathbb{G}_a) \underset{-\otimes_{k[\eta]} k}{\overset{(-)^{\mathbb{G}_a}}{\rightleftarrows}} \mathrm{Mod}(k[\eta])$$

  Note that the augmentation module $k \in \mathrm{Mod}(k[\eta])$ is non-compact and goes to the infinite-dimensional cofree $\mathbb{G}_a$-representation.

- If G is coaffine, then it is affine by definition.[3] For example, for $G = B\mathbb{G}_a$ we have the usual Koszul duality, first identifying $\mathrm{Rep}(B\mathbb{G}_a)$ with comodules for $\mathcal{O}(B\mathbb{G}_a)$, then with modules for its k-linear dual $\mathcal{O}(B\mathbb{G}_a)^* \simeq k[\lambda]$:

$$\mathrm{Rep}(B\mathbb{G}_a) \simeq \mathrm{Mod}(k[\lambda]) \underset{-\otimes_{k[u]} k}{\overset{(-)^{B\mathbb{G}_a}}{\rightleftarrows}} \mathrm{Mod}(k[u])$$

  where $k[u] \simeq \mathcal{O}(B^2\mathbb{G}_a)$. Note the above isn't quite right, we have to renormalize.

- If $G = S^1$, obviously this is not affine. Its affinization is $B\mathbb{G}_a$.

- If $G = E$, it also not affine, and its affinization is again $B\mathbb{G}_a$. Indeed, if V is a representation of E, then the action map must have proper image, therefore 0-dimensional image. But it must also be connected, so it is the identity.

[2] This isn't precisely defined, of course. We leave it open to interpretation.

[3] We allow ourselves some flexibility with what this means, e.g. the Spec vs. cSpec.

2 1-categorical

2.0.1. We now discuss higher representations and sheaves.

Definition 2.0.2. For any group object G, the category QCoh(G) is a comonoidal category. We define the 2-category of *2-representations*:

$$2\mathrm{Rep}(G) := \mathbf{Comod}_{\mathbf{dgCat}_k}(\mathrm{QCoh}(G)).$$

We also define for any prestack X the 2-category of *2-quasicoherent sheaves* on X to be the category of sheaves of categories on X.

$$2\mathrm{QCoh}(X) := \mathrm{ShCat}(X).$$

We say that X is *1-affine* if $2\mathrm{QCoh}(X) \simeq \mathbf{Mod}(\mathrm{QCoh}(X))$. The *1-affinization*[4] $1\mathrm{Aff}(X)$ of X is a 1-affine Y with a map $X \to Y$ defining an equivalence $2\mathrm{QCoh}(X) \simeq \mathbf{Mod}(\mathrm{QCoh}(Y))$.

2.0.3. Assuming that G is 1-affine,[5] we have

$$2\mathrm{Rep}(G) \simeq 2\mathrm{QCoh}(BG).$$

Assuming that BG is 1-affine, we then have an equivariantization correspondence:

$$2\mathrm{Rep}(G) \simeq \mathbf{Mod}(\mathrm{QCoh}(BG)).$$

2.0.4. Examples.

1. For G affine algebraic, the correspondence is well-known. For example, one can recover via descent that

   $$\mathrm{QCoh}(X/G) \otimes_{\mathrm{QCoh}(BG)} \mathbf{Vect}_k \simeq \mathrm{QCoh}(X)$$

   with the usual QCoh(G)-action.

2. As discussed earlier, $B^2\mathbb{G}_a$ is 0-affine, so it is 1-affine, so it $B\mathbb{G}_a$ satisfies 1-equivariantization.

3. On the other hand, S^1 satisfied 0-equivariantization for the dumb reason that $\mathrm{Rep}(S^1)$ is basically $\mathrm{Rep}(B\mathbb{G}_a)$, even though BS^1 was not 0-affine. However, S^1 does not satisfy 1-equivariantization. For example, consider the regular representation $\mathrm{QCoh}(S^1) \simeq \mathrm{QCoh}(\mathbb{G}_m)$. We have

   $$(\mathrm{QCoh}(S^1)^{\mathrm{QCoh}(S^1)} \otimes_{\mathrm{QCoh}(BS^1)} \mathbf{Vect}_k \simeq \mathbf{Vect}_k \otimes_{\mathrm{QCoh}(BS^1)} \mathbf{Vect}_k \simeq \mathbf{Vect}_k \otimes_{\mathrm{QCoh}(B^2\mathbb{G}_a)} \mathbf{Vect}_k \simeq \mathrm{QCoh}(B\mathbb{G}_a).$$

   This is the full subcategory of $\mathrm{QCoh}(S^1)$ where the automorphism is unipotent.

4. I have nothing intelligent to say about the elliptic situation.

3 Examples for S^1 and $B\mathbb{G}_a$ actions

3.0.1. Let's try to see the phenomenon in these examples. First, Cartier duality.

Theorem 3.0.2. *There is an equivalence of monoidal categories*

$$(\mathrm{QCoh}(S^1), \circ) \simeq (\mathrm{QCoh}(\mathbb{G}_m), \otimes).$$

$$(\mathrm{QCoh}(B\mathbb{G}_a), \circ) \simeq (\mathrm{QCoh}(\widehat{\mathbb{G}}_a), \otimes).$$

Invariants is identified with the !-fiber at 1, and coinvariants with the $*$-fiber.

[4] I am not sure about existence nor uniqueness.
[5] I think this can be relaxed, but I'm not entirely sure how.

3.0.3. So, a category with an S^1-action sheafifies over $\mathbb{G}_m$, and a category with a $B\mathbb{G}_a$-action is one supported at $1 \in \mathbb{G}_m$.

3.0.4. But we can do even more. Let us take the following general set-up. Let $\mathbf{X}$ be an integer lattice, $T = B\mathbf{X}$ the corresponding topological torus, $\check{T} = \operatorname{Spec} k\mathbf{X}$ the dual algebraic torus, $\mathfrak{t} = \mathbf{X} \otimes_{\mathbb{Z}} k$ the Lie algebra, and $\check{\mathfrak{t}}$ its dual. In fact, a category with an S^1-action, or a category over $\check{T}$, naturally sheafifies over "2-shifted" version of $\mathbb{T}^*_{\check{T}}$, namely $\check{T} \times \mathfrak{t}[2]$, and with a $B\mathfrak{t}$-action sheafifies over $\mathfrak{t}[2]$.

3.0.5. An important example: We can define an S^1-action on $\operatorname{Coh}(X)$ by a map $X \to \mathbb{G}_m$. Then, $\operatorname{Coh}(X)$ is a sheaf of categories over $\mathbb{G}_m \times \mathbb{A}^1[2]$. Given a $t \in \mathbb{G}_m$, we have:

$$\operatorname{Coh}(X)|_{\{z\}\times\mathbb{A}^1[2]} = \operatorname{Coh}(f^{-1}(z))$$

$$\operatorname{Coh}(X)|_{\{z\}\times\mathbb{G}_m[2]} = \operatorname{MF}(X, f-1)$$

$$\operatorname{Coh}(X)|_{\{z\}\times\widehat{\{0\}}} = \operatorname{Perf}(f^{-1}(z))$$

$$\operatorname{Coh}(X)|_{\widehat{\{z\}}\times\{0\}} = \operatorname{Coh}_{f^{-1}(z)}(X)$$

Let's justify some of these, assuming X is smooth for simplicity. The first one is "obvious", i.e. $\operatorname{Coh}(X) \otimes_{\operatorname{Perf}(\mathbb{G}_m)} \operatorname{Perf}(\{z\}) = \operatorname{Coh}(f^{-1}(z))$. There is a $k[u]$-module structure on this category. The u-torions are just the perfect complexes, and that tells us the second and third. For the fourth, we want to set $u = 0$, and you get the orthogonal Lagrangian.

3.0.6. Not every S^1-action on $\operatorname{Coh}(X)$ comes from viewing X over $\mathbb{G}_m$ when X is a stack or derived scheme. For example, there is an S^1-action on $\operatorname{Coh}(\mathcal{L}X)$. When X is a scheme, this is the de Rham differential (in fact, $\operatorname{Coh}(\mathcal{L}X)$ lives over $\widehat{\{1\}}$, i.e. gets a $B\mathbb{G}_a$-action). When $X = BG$, it is the tautological automorphism on adjoint-equivariant bundles on G.

3.0.7. Question: what is $\operatorname{Coh}(\mathcal{L}(BG)) \otimes_{\mathcal{O}(\mathbb{G}_m)} \{1\}$? Is it $\operatorname{Coh}(\widehat{\mathcal{L}}(BG))$?

4 Elliptic curves E

4.0.1. I know basically nothing about this. See recent work of Sibilla–Tomasini, which mentions work of Grojnowski.